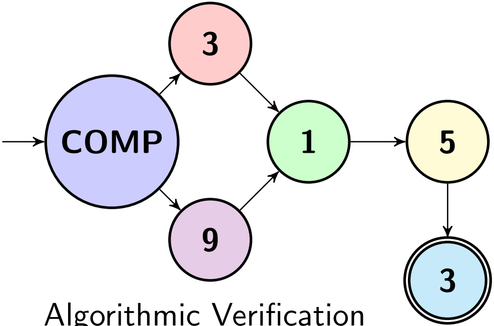

Algorithmic Verification

Expressiveness, CTL Model Checking

Dr. Liam O'Connor
CSE, UNSW (for now)
Term 1 2020

Comparing Logics

Formula Equivalence

Two formulae are equivalent iff they admit the same models.

$$\frac{\forall A.\ (A \models P) \Leftrightarrow (A \models Q)}{P \equiv Q}$$

Logic Expressiveness

A logic L_1 is *more expressive* than a logic L_2, written $L_2 \subseteq L_1$, iff:

For all $\varphi_2 \in L_2$, *there is a* $\varphi_1 \in L_1$ *such that* $\varphi_1 \equiv \varphi_2$.

CTL $\subseteq$ CTL*? LTL $\subseteq$ CTL*? *LTL* $\subseteq$ CTL? *CTL* $\subseteq$ LTL?

LTL $\subseteq$ CTL*

LTL formulae look like CTL* *path formulae*. How do we convert them into equivalent *state formulae*?

Recall that $A \models \varphi$ iff $\forall \rho \in \text{Traces}(A).\ \rho \models \varphi$

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LTL formulae look like CTL* *path formulae*. How do we convert them into equivalent *state formulae*?

Recall that $A \models \varphi$ iff $\forall \rho \in \mathrm{Traces}(A).\ \rho \models \varphi$

For all LTL formulae φ:

$$A \models_{\mathrm{LTL}} \varphi \iff A \models_{\mathrm{CTL}^*} \mathbf{A}\ \varphi$$

Proof follows trivially from the definition of **A**.

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CTL Formula: **AF AG** ●

CTL ⊆ LTL?

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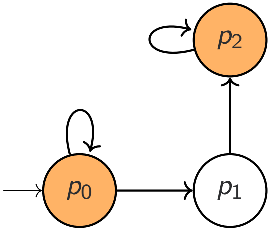
LTL Formula: **FG** ●? does this work?



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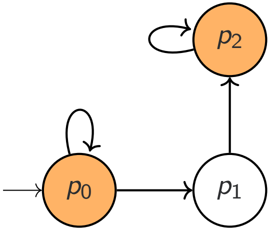

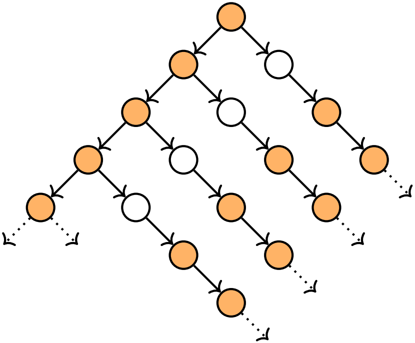

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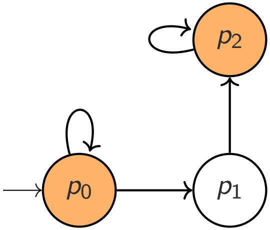
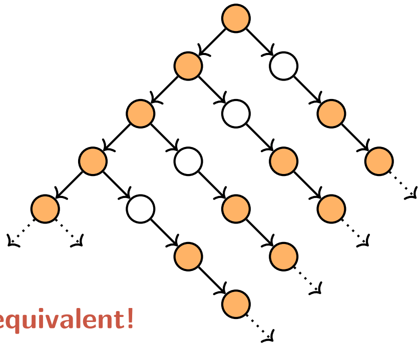


It's not equivalent!



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If $\text{Traces}(A) \subseteq \text{Traces}(\text{B})$ then for any LTL formula φ,
$B \models \varphi \implies A \models \varphi$

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Suppose $\exists$ an LTL formula φ that is equivalent to **AG EF** ●.

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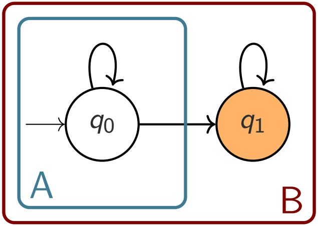

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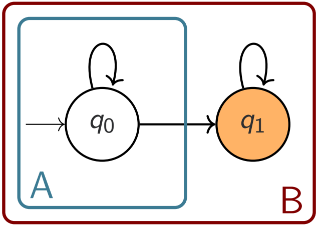

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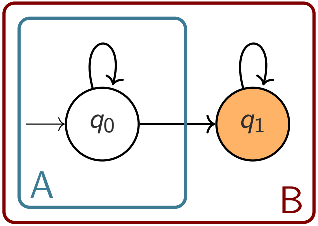

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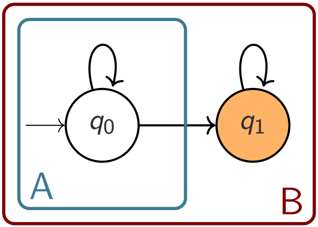

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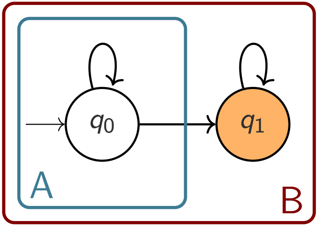

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Contradiction!

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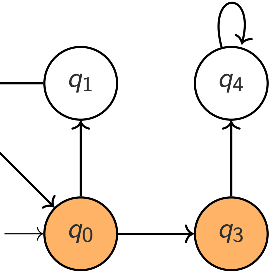

Nope!

LTL $\not\subseteq$ CTL

Lemma

It is possible to construct two families of automata A_i and B_i such that:

- They are distinguished by the LTL formula **F G** ●, that is: $A_i \models$ **F G** ● but $B_i \not\models$ **F G** ● for any i.

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- They cannot be distinguished by CTL formulae of length $\leq i$. That is, $\forall i.\ \forall \varphi.\ |\varphi| \leq i \Rightarrow (A_i \models \varphi \Leftrightarrow B_i \models \varphi)$

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Let φ be a CTL formula equivalent to **F G** ●.

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Yes. We know already that LTL $\subseteq$ CTL* and that LTL $\not\subseteq$ CTL. So pick any LTL formula that cannot be expressed in CTL, and we have a formula that cannot be expressed in CTL but can be in CTL*.

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Yes. The proof is very involved, but the formula **E G F** ● cannot be expressed in either LTL nor CTL.

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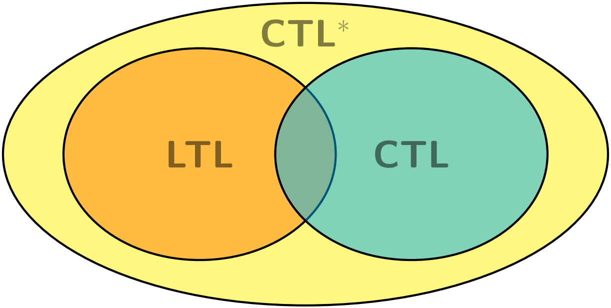


The CTL Model Checking Problem

Given

- A CTL formula φ, and
- An automaton A,

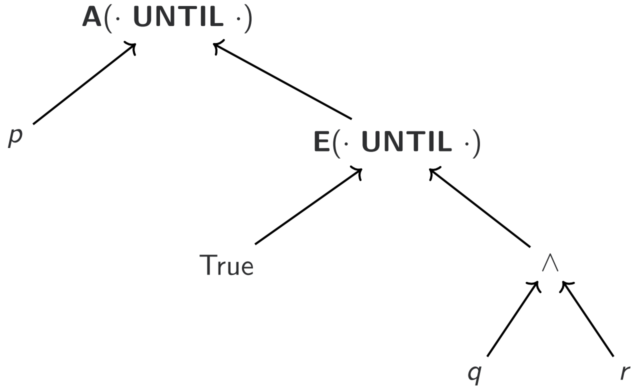
Determine if $A \models \varphi$.

Our approach

We first break the formula up into a *parse tree*. Then, annotate states in a bottom-up fashion with the (sub-)formulae they satisfy.

Parse Trees

$$\mathbf{A}(p \ \mathbf{UNTIL} \ \mathbf{E}(\text{True} \ \mathbf{UNTIL} \ q \wedge r))$$



Formal Algorithm: Basic Propositions

```
case φ ∈ 𝒫 do                                   /* Atomic proposition */
    foreach q ∈ Q do
        if φ ∈ L(q) then
            q.φ := True;
        else
            q.φ := False;

case φ = ¬ψ do                                  /* Negation */
    Mark(A, ψ) ;
    foreach q ∈ Q do
        q.φ := ¬q.ψ ;

case φ = ψ1 ∧ ψ2 do                             /* Conjunction */
    Mark(A, ψ1); Mark(A, ψ2) ;
    foreach q ∈ Q do
        q.φ := q.ψ1 ∧ q.ψ2 ;
```

Formal Algorithm: EX

```
case φ = EX ψ do                         /* Exists a Successor */
    Mark(A, ψ) ;
    foreach q ∈ Q do
        q.φ := False;
    foreach (q, q') ∈ δ do
        if q'.ψ then
            q.φ := True ;
```

We can simplify **AX** ψ to $\neg$**EX** $\neg\psi$. Why?

```
case φ = E ψ1 UNTIL ψ2 do                          /* Exist Until */
  Mark(A, ψ1) ; Mark(A, ψ2) ;
  foreach q ∈ Q do
    q.φ := False;
    q.visited := False;
    if q.ψ2 then
      q.φ := True ;
      q.visited := True ;
      W := W ∪ {q};
  while W ≠ ∅ do
    q := pop(W);  /* q satisfies φ */
    foreach (q′, q) ∈ δ do
      if ¬q′.visited then
        q′.visited := True ;
        if q′.ψ1 then
          q′.φ := True; W := W ∪ {q′};
```

```
case φ = A ψ1 UNTIL ψ2 do                    /* For All Until */
    Mark(A, ψ1) ; Mark(A, ψ2);
    foreach q ∈ Q do
        q.φ := False;
        q.nbUnchecked := |δ(q)|;
        if q.ψ2 then
            q.φ := True ;
            W := W ∪ {q};
    while W ≠ ∅ do
        q := pop(W);
        /* q satisfies φ */
        foreach (q′, q) ∈ δ do
            q′.nbUnchecked := q′.nbUnchecked − 1 ;
            if (q′.nbUnchecked = 0 ∧ q′.ψ1 ∧ ¬q′.φ) then
                q′.φ := True ;
                W := W ∪ {q′};
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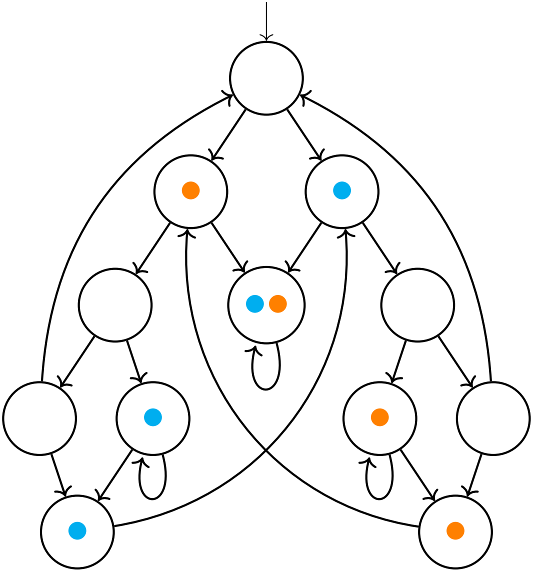
Therefore, overall complexity is: $\mathcal{O}(\ (|Q| + |\delta|) \times |\varphi|\ )$

Example



Procedure

- **Simplify** to basic CTL operations.
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Example

- **EF** $(\bullet \wedge \bullet)$

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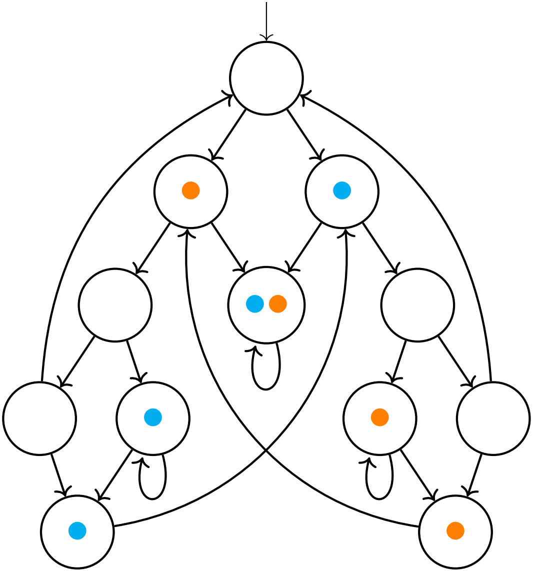

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- **EF** (● ∧ ●)
- **EF AG** (● ∧ ●)

Bibliography

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CTL Model Checking

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